Sections 2.6 & 3.6

Derivatives and Limits of Trigonometric Functions

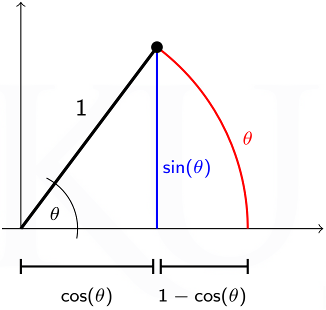
(1) Trigonometric Limit Identities

(2) Derivatives of Trigonometric Functions

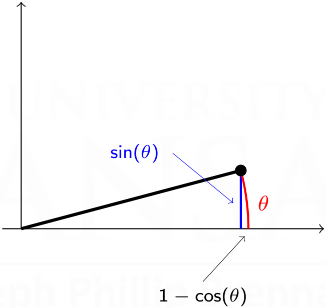
(3) Derivatives of Inverse Trigonometric Functions

Trigonometric Limits

Definition of sine and cosine:



When θ is very small:



Two Key Trigonometric Limits

$$\lim_{x \to 0} \frac{\sin(x)}{x} = 1 \qquad \lim_{x \to 0} \frac{\cos(x) - 1}{x} = 0$$

Derivative of the Sine and Cosine Functions

$$\frac{d}{dx}(\sin(x)) = \overbrace{\lim_{h\to 0}\frac{\sin(x+h)-\sin(x)}{h}}^{\text{The definition of derivative}}$$

$$= \lim_{h\to 0}\frac{\overbrace{\sin(x)\cos(h)+\cos(x)\sin(h)}^{\text{Sine of a sum formula}}-\sin(x)}{h}$$

$$= \sin(x)\boxed{\left(\lim_{h\to 0}\frac{\cos(h)-1}{h}\right)}+\cos(x)\boxed{\left(\lim_{h\to 0}\frac{\sin(h)}{h}\right)}$$

$$= \cos(x).$$

A similar calculation shows that $\frac{d}{dx}(\cos(x)) = -\sin(x)$.

Derivatives of Trigonometric Functions

$$\frac{d}{dx}\big(\sin(x)\big) = \cos(x) \qquad \frac{d}{dx}\big(\cos(x)\big) = -\sin(x)$$

$$\frac{d}{dx}\big(\tan(x)\big) = \sec^2(x) \qquad \frac{d}{dx}\big(\cot(x)\big) = -\csc^2(x)$$

$$\frac{d}{dx}\big(\sec(x)\big) = \sec(x)\tan(x) \qquad \frac{d}{dx}\big(\csc(x)\big) = -\csc(x)\cot(x)$$

These identities can be obtained from the derivatives of $\sin(x)$ and $\cos(x)$, using the definitions of the other trigonometric functions and the identity $\sin^2(x) + \cos^2(x) = 1$.

Derivative of the Tangent Function

$$\frac{d}{dx}\left(\tan(x)\right) = \frac{d}{dx}\left(\frac{\sin(x)}{\cos(x)}\right)$$

Use the Quotient Rule:

$$= \frac{\cos(x)\left(\frac{d}{dx}\left(\sin(x)\right)\right) - \sin(x)\left(\frac{d}{dx}\left(\cos(x)\right)\right)}{\cos^2(x)}$$

$$= \frac{\cos^2(x) + \sin^2(x)}{\cos^2(x)} = \frac{1}{\cos^2(x)} = \boxed{\sec^2(x)}$$

Derivatives of Other Trigonometric Functions

$$\frac{d}{dx}\big(\sec(x)\big) = \frac{d}{dx}\big(\cos(x)^{-1}\big)$$

Use the Chain Rule:

$$= -(\cos(x)^{-2}) \cdot \frac{d}{dx}\big(\cos(x)\big)$$

$$= \frac{-\sin(x)}{-\cos^2(x)} = \left(\frac{1}{\cos(x)}\right)\left(\frac{\sin(x)}{\cos(x)}\right) = \boxed{\sec(x)\tan(x)}$$

The derivatives of $\csc(x) = \dfrac{1}{\sin(x)}$ and $\cot(x) = \dfrac{1}{\tan(x)}$ can be calculated in similar ways.

Example 1

(I) $\dfrac{d}{dx}\left(\cos\left(e^{\tan(x)}\right)\right) = -\sin\left(e^{\tan(x)}\right) e^{\tan(x)} \sec^2(x)$

(II) $\dfrac{d}{dx}\left(\sec^2(4x)\right) = 8\sec^2(4x)\tan(4x)$

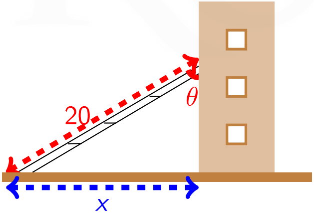
(III) $\dfrac{d}{dx}\left(\sin^2(4x) + \cos^2(4x)\right) = 0$

(IV) $\dfrac{d^2}{dx^2}\left(\sin(x)\right) = -\sin(x)$

Example 2

A ladder is 20 feet long and rests against a vertical wall. Let θ be the angle between the top of the ladder and the wall. If the bottom of the ladder slides away from the wall so that the angle increases at a rate of 1 radian/second, how fast is the bottom of the ladder moving when $\theta = \frac{\pi}{3}$?

Let x be the distance from the base of the ladder to the wall.



$$x = 20\sin(\theta)$$

$$\frac{dx}{dt} = \frac{dx}{d\theta}\frac{d\theta}{dt} = (20\ \cos(\theta))\,(1\text{ rad/sec})$$

$$\left.\frac{dx}{dt}\right|_{\theta=\pi/3} = (10\text{ ft/rad})(1\text{ rad/sec}) = \boxed{10\text{ ft/sec.}}$$

Trigonometric Limits

Example 3

$$\lim_{x\to 0}\frac{\sin(4x)}{x} = \lim_{x\to 0}\frac{4\sin(4x)}{4x}$$

$$= \left[\lim_{x\to 0} 4\left(\frac{\sin(4x)}{4x}\right)\right]$$

$$= 4\lim_{z\to 0}\frac{\sin(z)}{z}$$

$$= 4.$$

More Trigonometric Limits

Example 4

$$\lim_{x\to -2} \frac{\sin(x+2)}{x^2+x-2} = \lim_{x\to -2} \frac{\sin(x+2)}{(x+2)(x-1)}$$

$$= \left[\lim_{x\to -2} \left(\frac{1}{x-1}\right)\right] \left[\lim_{x\to -2} \frac{\sin(x+2)}{x+2}\right]$$

$$= -\frac{1}{3} \lim_{z\to 0} \frac{\sin(z)}{z}$$

$$= -\frac{1}{3}.$$

More Trigonometric Limits

Example 5

$$\lim_{x\to 0} \frac{1-\cos(3x)}{x^2} = \lim_{x\to 0}\left[\frac{1-\cos(3x)}{x^2}\times\frac{1+\cos(3x)}{1+\cos(3x)}\right]$$

$$= \lim_{x\to 0}\left(\frac{1-\cos^2(3x)}{x^2\left(1+\cos(3x)\right)}\right) = \lim_{x\to 0}\frac{\sin^2(3x)}{x^2\left(1+\cos(3x)\right)}$$

$$= \left[\lim_{x\to 0}\frac{3\sin(3x)}{3x}\right]\left[\lim_{x\to 0}\frac{3\sin(3x)}{3x}\right]\left[\lim_{x\to 0}\frac{1}{(1+\cos(3x))}\right]$$

$$= (3)(3)\left(\frac{1}{2}\right) = \frac{9}{2}.$$

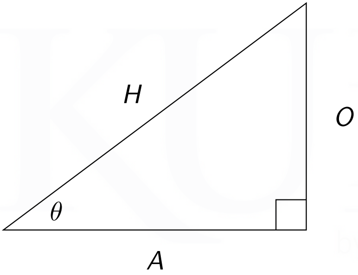
More Trigonometric Limits

Example 6

$$\begin{aligned}
\lim_{x\to 0} \frac{\tan^2(3x)}{x^2} &= \lim_{x\to 0} \left(\frac{\sin^2(3x)}{x^2 \cos^2(3x)} \right) \\
&= \left[\lim_{x\to 0} \frac{1}{\cos^2(3x)} \right] \left[\lim_{x\to 0} \frac{\sin^2(3x)}{x^2} \right] \\
&= 1 \cdot \lim_{x\to 0} \frac{9 \sin^2(3x)}{(3x)^2} = 9 \left(\lim_{x\to 0} \frac{\sin(3x)}{3x} \right)^2 \\
&= 9.
\end{aligned}$$

Derivatives of Inverse Trigonometric Functions

The **arcsine**, **arccosine** and **arctangent** functions are defined as inverses of the basic trigonometric functions.



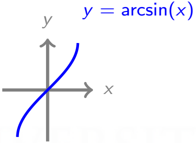
$$\sin(\theta) = O/H \qquad \arcsin(O/H) = \theta$$
$$\cos(\theta) = A/H \qquad \arccos(A/H) = \theta$$
$$\tan(\theta) = O/A \qquad \arctan(O/A) = \theta$$

Warning: Do not confuse $\arcsin(\theta)$ with $\frac{1}{\sin(\theta)} = \csc(\theta)$!

The graphs of $\sin(x)$, $\cos(x)$ and $\tan(x)$ do not pass the Horizontal Line Test. Therefore, in order to define inverse functions, we must restrict their domains.

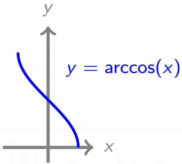
Arcsine

- Domain: $-1 \leq x \leq 1$
- Range: $\frac{-\pi}{2} \leq y \leq \frac{\pi}{2}$



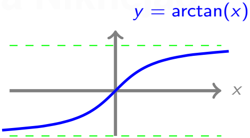
Arccosine

- Domain: $-1 \leq x \leq 1$
- Range: $0 \leq y \leq \pi$

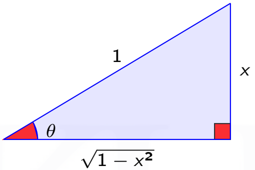


Arctangent

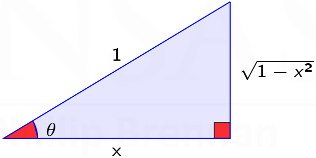
- Domain: $-\infty < x < \infty$
- Range: $\frac{-\pi}{2} < y < \frac{\pi}{2}$



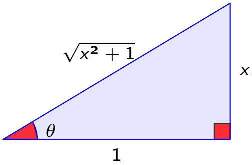
Precalculus Review



- $\sin\left(\arcsin(x)\right) = x$
- $\cos\left(\arcsin(x)\right) = \sqrt{1 - x^2}$
- $\tan\left(\arcsin(x)\right) = \dfrac{x}{\sqrt{1 - x^2}}$



- $\sin\left(\arccos(x)\right) = \sqrt{1 - x^2}$
- $\cos\left(\arccos(x)\right) = x$
- $\tan\left(\arccos(x)\right) = \dfrac{\sqrt{1 - x^2}}{x}$



- $\sin\left(\arctan(x)\right) = \dfrac{x}{\sqrt{x^2 + 1}}$
- $\cos\left(\arctan(x)\right) = \dfrac{1}{\sqrt{x^2 + 1}}$
- $\tan\left(\arctan(x)\right) = x$

Derivatives of Inverse Trigonometric Functions

We can find the derivative of $y = \arcsin(x)$ by implicit differentiation.

$$y = \arcsin(x)$$

$$\sin(y) = x \qquad \text{Differentiate both sides:}$$

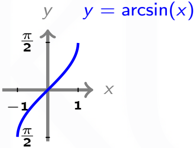
$$\cos(y)y' = \frac{d}{dx}(x) = 1$$

$$y' = \frac{1}{\cos(y)} = \frac{1}{\cos(\arcsin(x))} = \frac{1}{\sqrt{1-x^2}}$$

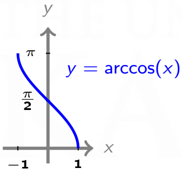
$$\frac{d}{dx}(\arcsin(x)) = \frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}(\arctan(x)) = \frac{1}{1+x^2}$$

$$\frac{d}{dx}(\arccos(x)) = \frac{-1}{\sqrt{1-x^2}}$$

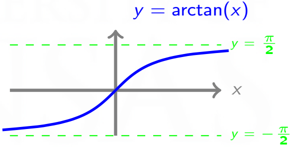
Derivatives of Inverse Trigonometric Functions



$$\frac{d}{dx}(\arcsin(x)) = \frac{1}{\sqrt{1-x^2}}$$



$$\frac{d}{dx}(\arccos(x)) = \frac{-1}{\sqrt{1-x^2}}$$



$$\frac{d}{dx}(\arctan(x)) = \frac{1}{1+x^2}$$